Precession of geodesics in space section of Schwarzschild's spacetime

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January 22, 2020

Summary The precession, resulting from the 4D spacetime curvature, of spacetime geodesics in Schwarzschild's spacetime, is well-known topic.[1]. The geodesic curve in this spacetime is fully defined by a $r(\varphi)$ function on a curve defined in 2D space. In this article, after a review of spacetime geodesics and of their space sections, we will derive the precession of a geodesic in the space section of the Schwarzschild's spacetime. Even though a space section of a spacetime has no physical character, it might enlightens the understanding of the geometry of the Schwarzschild's spacetime. This precession in this space section, which, unlike the precession in spacetime, will only depends on the ratio M/l, is defined by an original infinite polynomial of even powers of M/l, this providing a real precession, even for imaginary values of l^2 . At the end, we will review, the possible relation between this precession of geodesics in a space section and that of spacetime geodesics, in the case of weak field, within some assumptions, as well as its relation with the deflection of light by massive bodies (null geodesics). In the conclusion, we will discuss how this solution may enlighten the understanding of the precession phenomenology.

1 Spacetime and space geodesics in Schwarzschild's metric

For defining the precession of the perihelion of a geodesic in Schwarzschild's 4D spacetime, a function $r(\varphi)$, depending on parameters M (mass of the central body) and on the parameters of the geodesics :$L = r^2 d\varphi/d\tau$ (angular momentum) and $E = (1-2GM/r)dt/d\tau$ (energy), is only needed. The geodesic is a curve in spacetime (see figure 1 in annex 1), where per the spherical symmetry, the space section of the geodesic is a curve included in a 2D plane ($\theta = constant$, usually one set $\theta = \pi/2$)[3]. In the three dimensional space section of Schwarzschild's spacetime where $d\sigma^2$ is the metric line element, we can also to define the precession of a geodesic in this space section by a $r(\varphi)$ function, with an angular momentum defined by $l = r^2 d\varphi/d\sigma$. [4]

Original Schwarzschild's metric is recalled in equation (1) below.

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 \ + \sin^2\theta d\varphi^2) = -(1 - \frac{2GM}{r})dt^2 + d\sigma^2 \tag{1}$$

Where,

$$d\sigma^2 = +\frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 \ + \sin^2\theta d\varphi^2) = \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\varphi^2, \text{ for } \theta = \frac{\pi}{2} \tag{2}$$

In the last part of the equation , we set $\theta = \pi/2$, which is allowed, per the spherical symmetry, without loss of generality.

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[1]For analytic solution solving the spacetime equation, see for instance [**?**], chapter 19.

[2]M is the mass of the central body and l is a generic notation for the angular momentum in spacetime or in space

[3]This curve which is the projection of the spacetime geodesic on this plane is not a geodesic in space, of affine parameter σ used in the line element $d\sigma^2$ of the space metric, see annex 1.

[4]In 4D spacetime, the affine (dynamic) parameter on a timelike geodesic is the proper time, a timelike parameter. In space the affine parameter on the spacelike geodesic, is a spacelike parameter therefore the dimension of $L = r^2 d\varphi/d\tau$ will be a square length divided by a time while that of $l = r^2 d\varphi/d\sigma$ will be a length.

2 Analytic method for the precession of planets in Schwarzschild's spacetime

2.1 Interest of geodesics in a space section of spacetime

General relativity is a geometrical theory of the gravitation in spacetime. Physical geodesics are timelike or null geodesics. [5] Unlike geodesics in spacetime which do not depend on the coordinates, geodesics in a space section of spacetime [6] depend on the coordinates. So, what would be the interest of such geodesics? We may select a space section in spacetime, but we know that this depends on the selected coordinates, and that this space section (at constant time in the coordinates attached at this frame), is arbitrary [7]. This interest is motivated by an original proposal of Painlevé [?] describing the Schwarzschild's spacetime geodesic as the geodesic of the Schwarzchild space section multiplied by a conformal factor. This shows that the conformal structure of the space section and that of the spacetime are the same. This is a quite surprising property. See annex 1 for some details.

But, even though a space section is arbitrary, the space section in the coordinates used in the Schwarzschild frame as described in equation (1) may be interesting as on the one hand the time is "'orthogonal"' to the space section [8] and on the other hand the form of the metric is "'static"'. [9] Therefore, as one defines usually the spacetime geodesic by using the Schwarzschild coordinates, geodesics in its space section would provide a complementary information for describing the geometrical space structure of a space section which is more complex than it appears at a first look, see [?], figure 1.

2.2 The geodesic equation in space

We will use the well-known method for getting a general solution for the geodesic equation in space and then, we will use it for solving the problem of the precession of geodesics in space. Dividing second part of equation (2) by $d\sigma^2$ yields:

$$1 = \frac{dr^2}{d\sigma^2(1-\frac{2GM}{r})} + r^2\frac{d\varphi^2}{d\sigma^2} \Rightarrow \frac{dr^2}{d\sigma^2} = (1-\frac{2GM}{r})(1-r^2\frac{d\varphi^2}{d\sigma^2}) \tag{3}$$

$$\frac{dr^2}{d\sigma^2} = (1-\frac{2GM}{r})(1-\frac{l^2}{r^2}) \tag{4}$$

Equation (4), valid only on a geodesic, is equation (3) with $l = r^2 d\varphi/d\sigma$ which is the conserved angular momentum, on the spatial geodesic.[10]

By multiplying equation (4) by $(d\sigma/d\varphi)^2 = r^4/l^2$, we get:

$$\frac{dr^2}{d\varphi^2} = (1-\frac{2GM}{r})(\frac{r^4}{l^2}-r^2) \Rightarrow d\varphi = \frac{\pm dr}{\sqrt{-r^2(1-\frac{2GM}{r})(1-\frac{r^2}{l^2})}} \tag{5}$$

Let us set:

$$u = \frac{1}{r} \Rightarrow r = \frac{1}{u} \Rightarrow dr = -\frac{du}{u^2} \tag{6}$$

By inserting it, in equation (5), we get :

$$\frac{du^2}{u^4 d\varphi^2} = (1-2GMu)(\frac{1}{u^4})(\frac{1}{l^2}-u^2) \Rightarrow d\varphi = \frac{\pm du}{\sqrt{(1-2GMu)(\frac{1}{l^2}-u^2)}} \tag{7}$$

By defining an angle θ, a parameter A^2 and a constant K, such as:

$$\theta = \arcsin\sqrt{\frac{1+lu}{2}} \Rightarrow \sin^2\theta = \frac{1+lu}{2}, A^2 = \frac{4GM}{2GM+l}, K = \sqrt{\frac{l}{l+2GM}} \tag{8}$$

[5] One can also define spacelike geodesics in spacetime but there are not considered to be physical.

[6] Geodesics in a space section of the Schwarzschild 's spacetime is a special case, where the coordinate t is constant, of spacelike geodesics, where the affine parameter of the geodesic is spacelike in this spacetime but, where the coordinate time t is generally not constant. In this paper, we limit our main analysis to geodesics in the space section of spacetime.

[7] We will select a space section in the Schwarzschild's coordinates, but a space section in the Painlevé's coordinates, describing the same spacetime would return an Euclidean space section whose geodesics are straight lines!

[8] This means that the four basis vectors, associated to the coordinates are orthogonal according to the definition of orthogonality in relativity.

[9] This means that the space section does not depend on time and is orthogonal to space.

[10] This "'constant of motion"' l exists as the metric $d\sigma^2$ does not depend on φ.

Equation (7) can be written:[11]

$$\frac{d\varphi}{2} = K\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} \Rightarrow \frac{\varphi(\psi,A^2)}{2} = K\int_{\theta=0}^{\theta=\psi}\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} = K\,Elliptic_F(\psi,A^2) \tag{9}$$

Inserting the values of θ, K and A^2 defined in equation (8) yields:

$$\frac{\varphi}{2} = \sqrt{\frac{l}{l+2GM}}Elliptic_F[\arcsin(\sqrt{\frac{1}{2}(1+lu)}),\frac{4GM}{l+2GM}] \tag{10}$$

$Elliptic_F(\psi,A^2)$ or $F(\psi,A^2)$ in a short notation is the integral described in equation (9). This integral, called elliptic integral of the first kind, includes an argument A^2 called the parameter, A is called the modulus.[12] The parameter $\psi=\theta(u)$, called the amplitude, is, as shown in equation (9), the upper limit of integration of the angle θ defined in equation (8).

Returning to $r=1/u$, we get:

$$\frac{\varphi}{2} = \sqrt{\frac{l}{l+2GM}}Elliptic_F[\arcsin(\sqrt{\frac{1}{2}(1+\frac{l}{r})}),\frac{4GM}{l+2GM}] \tag{11}$$

2.3 The non trivial structure of the space section

In equation (10), in the $Elliptic_F$ integral, the constraint on the parameter $\theta=\arcsin\sqrt{x}$ implies $0\le x\le 1\rightarrow -1/l\le u\le 1/l$ and the constraint $A^2\le 1$, implies that $2GM\le l$

We know that the Schwarzschild's solution is not the maximally extended solution for this spacetime as it describes only two spacetime regions, denoted I and II, of the four spacetime regions, denoted I, II, III, IV, described, for instance, by the Kruskal's solution (see [?] figure 5.42 p. 226, for instance) . The negative value of u and r, as $r=1/u$, should be associated to the space sections of regions III and IV of the Kruskal solution.

This shows that, even though the Schwarzchild's solution does not describes all the spacetime regions, its space section solution describes all space sections of the spacetime. This is possible as there is no singularity for $r=0$ in the space metric and because the singularity at $r=2GM$ is not physical. As, per the definition of the $Elliptic_F$ integral, $A^2\le 1\rightarrow l\ge 2GM$, this implies, in addition, that in this problem, only space sections of regions I and IV (outside of this horizon) are involved.

2.4 Elliptic-K

$$K(A^2) = Elliptic_K(A^2) = Elliptic_F(\frac{\pi}{2},A^2) = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} \tag{12}$$

$Elliptic_K(A^2)$, also called $K(A^2)$ in a short notation, is a special case of $Elliptic_F$, where the upper limit ψ is equal to $\pi/2$. It is the complete elliptic integral of the first kind of Legendre and therefore has only one parameter (A^2). There exists an analytic definition of the integral $Elliptic_K(A^2)$ by an infinite polynomial of powers of A^2. This is this polynomial definition, given by equation (19), that we will use further, for calculating the precession.

With the values of K and A^2, defined in equation (8), by using the definition of $Elliptic_K(A^2)$, the equation (10) for $\psi=\pi/2$, can be written:

$$\frac{\varphi(\psi=\pi/2,A^2)}{2} = \sqrt{\frac{l}{l+2GM}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} = \sqrt{\frac{l}{l+2GM}}Elliptic_K(\frac{4GM}{l+2GM}) \tag{13}$$

Equation(13) that we will use for computing the precession, defines $\varphi/2$ and not φ therefore, we will have to multiply it by two for getting the result. Equation (13) shows that the solution in space only depends on the parameters M and l.

[11] By using the definition of θ, A^2 and K in equation (8), annex 1 shows that it is straightforward to verify that $du^2/[(1-2GMu)(-u^2+1/l^2)] = (2Kd\theta)^2/(1-A^2\sin^2\theta)$ whose integral is an elliptic integral of first kind.The binary operator $\pm$, in the last term of equation (7), is related to the orientation of the geodesic as there are two possible orientations. For the integration, we can select one direction, without loss of generality, that we will associate to the sign +.

[12] There are several formal notations, this being quite confusing. For instance, it is denoted $Elliptic_F(\psi,A)$ in WolframMathWorld but, in both notations, it is A^2 which is used in the computation of the integral. It is just two notations for the same object. This remark will also apply to the $Elliptic_K$ integral and Jacobi-Amp function, that we will use further.

2.5 Jacobi-Amp-function

The $Elliptic_F$ integral defined in equation (10) $\varphi/2K = F(\psi, A^2)$ has an inverse function called Jacobi-Amplitude function noted $am(\varphi/2K, A^2)$, such that $\psi = am(\varphi/2K, A^2)$. So per the definition of $\sin(\psi)$, where ψ is the upper limit of θ, and K in equation (8) we get [13]:

$$\sin^2\psi = \frac{1+lu}{2} = sn^2(\frac{\varphi}{2K}, A^2) \Rightarrow u = \frac{1}{l}(-1 + 2sn^2(\frac{\varphi}{2K}, A^2)) \tag{14}$$

This gives the function $u(\varphi)$. The function $r(\varphi)$ can be deduced by using the relation $r = 1/u$. An example given in annex 2 shows the curvature of the spce section, exhibited by the precession of the space geodesics, of the Schwarzschild's spacetime.

2.6 These elliptic integrals define the precession

$Ellipic_F, Elliptic_K$ and their inverse integrals exhibit two angles ψ and φ. Equation (13) shows that when θ varies from *0* to $\psi = \pi/2$, $\varphi/2K$ varies from *0* to $Elliptic_K(A^2)$. So for a half-pseudo-orbit where θ varies from *0* to π [14]:

$$\Delta(\frac{\varphi}{2}) = 2K Elliptic_K(A^2) - \pi \tag{15}$$

And equation (16) below will be the equation to be used for solving the problem for n full orbits.

$$\Delta\varphi = 4n(2K Elliptic_K(A^2) - \pi) \tag{16}$$

2.7 Class of equivalence

The equation (16) only depends on parameters K and A^2. This equation will provide an exact solution to the problem, if we know these parameters on the spacelike geodesic. By posing $2GM/c^2l = k^2$, parameters K and A^2, defined in equation (8), can be written:

$$K = (1 + \frac{2GM}{c^2l})^{-1/2} = (1+k^2)^{-1/2}, A^2 = (\frac{2GM}{c^2l})(\frac{2}{1+\frac{2GM}{c^2l}}) = (2k^2/(1+k^2)) \tag{17}$$

Therefore, equation (16) will only depend on the dimensionless parameter k^2. We will expect a solution as a function of k^2.

This parameter k^2 defines a class of spatial solutions.

3 Solution for the precession in space

3.1 General solution

By inserting the definition of A^2 and K, given in equation (17), in equation (13), giving the formal general solution, for $\psi = \pi/2$, we get:

$$\frac{\varphi}{2} = \sqrt{\frac{1}{1+k^2}} Elliptic_K(\frac{2k^2}{k^2+1}) \tag{18}$$

The integral $Elliptick_K(k)$ can be represented by an infinite polynomial: [15]

$$Elliptic_K(\frac{2k^2}{k^2+1}) = \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n!)}{2^{2n}(n!)^2}]^2(\frac{2k^2}{k^2+1})^n \equiv \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(\frac{2k^2}{k^2+1})^n \tag{19}$$

where n!! denotes the semi-factorial. By using equation(19), equation (18) becomes:

$$\frac{\varphi}{2} = \sqrt{\frac{1}{1+k^2}}(\frac{\pi}{2})\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(\frac{2k^2}{1+k^2})^n = \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(2k^2)^n(1+k^2)^{-\frac{1}{2}(2n+1)} \tag{20}$$

[13] In Jacobi elliptic functions, $sn(\varphi/2K, A^2) = sin(am(\varphi/2K, A^2))$, see WolframMathWorld, Jacobi elliptic functions.

[14] A half-orbit defines the dynamic as we assume the symmetry of the orbit for the precession.

[15] *http://mathworld.wolfram.com/CompleteEllipticIntegraloftheFirstKind.html*, equation (2). In terms of the Gauss hypergeometric function, $Elliptic_K = (\pi/2)_2F_1(1/2, 1/2, 1, 2k^2/(1+k^2))$. Let us recall that the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ is a solution of the second order homogeneous differential equation $z(1-z)d^2y/dz^2 + [c-(a+b+1)z]dy/dz - aby = 0$.

We replaced $-(n+1/2)$ by $-(1/2)(2n+1)$ which will be more convenient. Let us recall that $(1+k^2)^{-\frac{1}{2}(2n+1)}$ can also be developed in an infinite polynomial, as defined below:

$$(1+k^2)^\alpha = 1 + \sum_{j=1}^{j=\infty} \frac{(\alpha)(\alpha-1)..(\alpha-j+1)}{j!}(k^2)^j \tag{21}$$

For $-1 < k^2 < 1$ and where α is a real number, with $\alpha = -(1/2)(2n+1)$ in our problem.

By using these formulas, equation (20) becomes: [16]

$$\frac{\varphi}{2} = \frac{\pi}{2} \sum_{n=0}^{n=\infty} (\frac{(2n-1)!!}{(2n)!!})^2 (2k^2)^n (1 + \sum_{j=1}^{j=\infty} \frac{(2n+1)(2n+3)..(2n+1+2(j-1))}{(-2^j)j!}(k^2)^j) \tag{22}$$

The product of these two infinite polynomials is an infinite polynomial. For defining this polynomial we have to calculate each coefficient B_n of $(k^2)^n$ which, in the infinite polynomial defined in equation (22), will be the sum of the product of coefficients of the terms$(k^2)^i$ of the first polynomial with the coefficients of the terms $(k^2)^{n-i}$ of the second polynomial.

The result of this operation will define an infinite polynomial $P(k^2)$

$$P(k^2) = \frac{\pi}{2} \sum_{n=0}^{n=\infty} B_n (k^2)^n \tag{23}$$

Where in B_n, as the first factor of the sum is related to $(k^2)^i$, the second factor should be related to $(k^2)^{n-i}$, therefore in equation (22), we must set $n = i$, $j = (n-i)$. This yields: $(2i+1)(2i+3)..(2i+1+(2(n-i-1))) = (2i+1)(2i+3)..(2n-1)) = (2n-1)!!/(2i-1)!!$. Therefore B_n can be written:

$$B_n(k^2)^n = \sum_{i=0}^{i=n} (\frac{(2i-1)!!}{2i!!})^2 (2^i (k^2)^i)(\frac{(2n-1)!!}{(2i-1)!!(n-i)!(-2^{n-i})})(k^2)^{n-i} \tag{24}$$

By simplifying by $(2i-1)!!$ and using $(2i-1)!! = 2i!/(i!2^i)$, $(2i)!! = i!2^i$, $2i!/i!^2 = 2i!/(i!(2i-i)!) = \binom{2i}{i}$, $n!/i!(n-i)! = \binom{n}{i}$, this equation yields:

$$B_n = \frac{2n!}{n!^2}(2^{-2n}) \sum_{i=0}^{i=n} \frac{(-1)^{n-i}}{2^i} \binom{n}{i}\binom{2i}{i} = \frac{2n!}{n!^2}(-1^n)(2^{-2n}) \sum_{i=0}^{i=n} (-\frac{1}{2})^i \binom{n}{i}\binom{2i}{i} \tag{25}$$

3.2 The polynomial includes only even powers of (k^2)

We will demonstrate that equation (25) giving B_n is a hypergeometric series. Such series is defined by using the Gauss hypergeometric function ${}_2F_1(a; b; c, d)$. [17]

Let us set:

$$A(n) = \frac{2n!}{n!^2}(-1^n)(2^{-2n}) \tag{26}$$

Per the formal definition of the Gauss hypergeometric function:

$${}_2F_1(a; b; c, d) = \sum_{i=0}^{i=n} \frac{(a)_i (b)_i}{(c)_i} \frac{(d)^i}{i!} \tag{27}$$

Where the notation $(a)_i = a(a+1)..(a+i-1)$ is the Pochhammer symbol. If we set $a = -n$, $b = 1/2$, $c = 1$, $d = 2$, we get:

$(a)_i = (-n)(-n+1)..(-n+i-1) = (-1)^i(n)(n-1)..(n-i+1) = (-1)^i n!/(n-i)!$, $(b)_i = (1/2)(3/2)...(2i-1)/2 = (1/2)^i(2i-1)!! = (1/2)^i 2i!/(2^i i!)$, $(c)_i = (1)(2)...(i) = i!$, $d^i = 2^i$

Inserting these values in equation (27) yields:

$${}_2F_1(-n; 1/2; 1, 2) = \sum_{i=0}^{i=n} (\frac{(-1)^i(n!)}{(n-i)!})((\frac{1}{2})^i \frac{2i!}{2^i i!})(\frac{1}{i!})(\frac{2^i}{i!}) = \sum_{i=0}^{i=n} (-\frac{1}{2})^i \binom{n}{i}\binom{2i}{i}) = \frac{B_n}{A(n)} \tag{28}$$

[16] In the second sum, we will separate the factor 1/2 and the sign $-$ from the formula and will gather them in the factor $1/(-2^j)$ for simplifying the calculation.

[17] See mathworld.wolfram-hypergeometricFunction, equation (8) for the form of the series generated by such function, and the Pochhammer notation symbol $(a)_n$.

In the second sum of the equation above we used the relations: $n!/((n-i)!i!) = \binom{n}{i}$ and $2i!/(i!i!) = \binom{2i}{i}$.
This is the result that we expected!

For demonstrating that all terms $B_{2m+1}(k^2)^{2m+1}$ vanish, we need to use equation (16) of mathworld.wolfram-HypergeometricFunction, which gives an integral defining the hypergeometric function.

$$ {}_2F_1(a;b;c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}du. \tag{29} $$

By calculating this integral with our parameters ($a=-n, b=1/2, c=1, z=2$), we get:

$$ \int_0^1 u^{-1/2}(1-u)^{-1/2}(1-2u)^n du = \int_0^{0.5} u^{-1/2}(1-u)^{-1/2}(1-2u)^n du + \int_{0.5}^1 u^{-1/2}(1-u)^{-1/2}(1-2u)^n du \tag{30} $$

Let us define two values of u, $(0 \leq u \leq 1)$, u_1 and u_2 such that $u_1 = 1/2 + a$ and $u_2 = 1/2 - a$, where $a \leq (1/2)$. We get $u_1^{-1/2}(1-u_1)^{-1/2} = [(1/2+a)(1/2-a)]^{-1/2} = u_2^{-1/2}(1-u_2)^{-1/2} = [(1/2-a)(1/2+a)]^{-1/2}$ (symmetry around 1/2).

In $(1-2u)$, for $u = u_1$ we get $-2a$ and for $u = u_2$, we get $2a$. This, raised to power n, will give $(u_1)^n = (-2a)^n$ and $(u_2)^n = (2a)^n$ which are equal when n is even and are opposite when n is odd.

Therefore, as exhibited by equation(30), where the integral is split in two parts (from 0 to 1/2 and from 1/2 to 1), when n is odd ($n = 2m+1$), the two parts are opposite, the integral vanishes and when n is even ($n = 2m$) the two parts are equal, this integral does not vanish.

Therefore:

$$ P(k^2) = \sum_{n=0}^{n=\infty} B_{2n}(k^2)^{2n} \tag{31} $$

includes only even powers of k^2.

$$ \frac{\varphi}{2} = K.Elliptic_K(A^2) = P(k^2) = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(k^4)^n = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(\frac{(2GM)^2}{(cL)^2})^n \tag{32} $$

The form of this polynomial, describing the precession, shows that, even for an imaginary value of the angular momentum, a real number for φ and therefore for the precession is expected! This is important because, per the form of the metric described in equations (1) and (2), if we assume that the angular momentum, on the timelike geodesic in spacetime, is real, that on the spacelike geodesic in space will be imaginary. Per this property, both precession in spacetime and space will be real.

A numerical value of this polynomial up to $n = 5$ is provided in equation (33).

3.3 Numerical value of the polynomial

The polynomial $P(k^2)$ is given below up to $n = 10$. [18]

$$ P(k^2) = \frac{\pi}{2}(1 + \frac{3}{16}(k^4) + \frac{105}{1024}(k^8) + \frac{1155}{16384}(k^{12}) + \frac{225225}{4194304}(k^{16}) + \frac{2909907}{67108864}(k^{20})) \tag{33} $$

4 Conclusion

In this paper, we focused our analysis on the space geodesic which does not look to be the most important in the theory of general relativity which is a spacetime theory. But as the most important information that we get in physics about some physical or geometrical objects does not reside in the objects themselves but in their relations, this complementary analysis which provides a set additional relations between objects may induce a better understanding of the underlying physics described by the theory.

[18]The coefficients B_n are computed by using mathematica line of command : $Bn = FullSimplify[((2n)!/(n!)^2)(2^{-2n}((-1)^n)Sum[((-1/2)^j)Binomial[n,j]Binomial[2j,j],(j,0,n)]]$, for $n = 2, 4, 6, 8, 10$.

A Annex 1: Form of the geodesic equation

We have to check that:

$$\frac{d\varphi^2}{4} = \frac{K^2 d\theta^2}{1 - A^2 \sin^2\theta} \text{ for } \sin\theta = \sqrt{\frac{1+lu}{2}}, A^2 = \frac{4GM}{2GM+l}, K = \sqrt{\frac{l}{l+2GM}} \tag{34}$$

is equivalent to:

$$\frac{l^2 du^2}{(1-2GMu)(1-u^2l^2)} \tag{35}$$

By taking the derivative of $sin(\theta)$, defined in eq. (36), we get:

$$d\theta^2 \cos^2\theta = \frac{l^2 du^2}{8(1+lu)} \Rightarrow d\theta^2(1-\sin^2\theta) = \frac{l^2du^2}{8(1+lu)} \tag{36}$$

$$\frac{d\theta^2}{2}(1-lu) = \frac{l^2du^2}{8(1+lu)} \Rightarrow d\theta^2 = \frac{l^2du^2}{4(1-l^2u^2)} \tag{37}$$

Therefore

$$d\varphi^2 = \frac{4K^2d\theta^2}{1-A^2\sin^2\theta} = \frac{4l}{l+2GM}\frac{l^2(2GM+l)du^2}{4(1-l^2u^2)(2GM+l-2GM(1+lu))} \tag{38}$$

By simplifying, this equation we get:

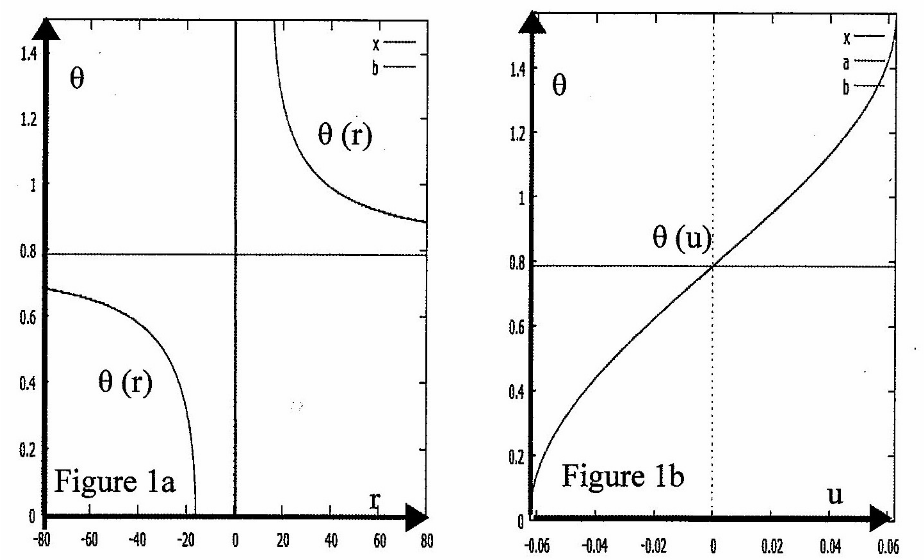
$$d\varphi^2 = \frac{l^2du^2}{(1-2GMu)(1-u^2l^2)} = \frac{du^2}{(1-2GMu)(\frac{1}{l^2}-u^2)} \Rightarrow d\varphi = \frac{du}{\sqrt{(1-2GMu)(\frac{1}{l^2}-u^2)}} \tag{39}$$

which is the equation (7), as expected.

B Annex 1: The formal tri-dimensional space

C Annex 1: The formal tri-dimensional space

C.1 Domain of definition of the Elliptic integrals in this space



The angular parameter $\theta(r) = arcsin(\sqrt{(1/2)(1+lu)}) = arcsin(\sqrt{(1/2)(1+l/r)})$, as $u = 1/r$, with its limit of integration, $\psi = \theta(u)$, the amplitude of the Elliptic integral, emerged in equations (14, 15).

The $Elliptic_F$ integral involves $\theta(u)$ where for $-1/l \leq u \leq 1/l$, we have : $0 \leq \theta(u) \leq \pi/2$. This was needed for using the $Elliptic_K$ integral.

This is described on figure 1b. Figure 1a describes $\theta(r)$. Both are drawn for $l = 16$. [19].

C.2 Domain of definition of the Elliptic integrals in this space

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C.3 Symmetries and topology of the formal tri-dimensional space

In chapter 4, in the demonstration, some intermediate results, got from formal calculation in a formal non physical 3D space [21], may include some negative values of the product $l.u$ or l/r where r is a coordinate (as well as $u = 1/r$). This suggest that u and r may have both positive and negative values in the equation. In [?], in figure 1, the topology of the 3-dimensional maximally extended Schwarzschild's metric is represented as two asymptotically flat space sections connected by a Rosen-Einstein bridge, therefore positive values of r may be associated to one of these space section (the usual physical space) and negative value to the other (its symmetrical counterpart in the maximally extended solution).

We made no assumption on the parameter l, but when comparing l (angular momentum in space) with L, its counterpart in spacetime, we will use the relation $l = iL$. The imaginary number i arise because per their definition, $l = r^2 d\varphi/d\sigma$ and $L = r^2 d\varphi/d\tau$, the dynamic parameter is timelike in spacetime and spacelike in space. But this is conventional, we might as well define l real and L imaginary. Anyway, we demonstrate that, even with imaginary (positive or negative) parameters, the equation (29) will return a real number for the precession.

The formal solution given by equation (14) yields $\varphi/2$.[22] If we assume a simplex topology for the group of rotation of parameter ψ supposed to be continuous parameter in this isotropic 3-dimensional extended space, $SU(2)$, where a rotation of 4π is needed for returning to initial position, must be the group to be used. In this 3-dimensional extended space this can be interpreted by its topology made of two folios. The angular parameter ψ spans the extended space while φ spans only the positive folio. This would explain that the equation involves $\varphi/2$.

Therefore, in space this why we use: $\Delta\varphi = \varphi(2\pi, A^2) - (-4\pi)$

C.4 Exhibition of the geodesic precession in space

The geodesic in space is a formal representation of the solution of the phenomenology of precession in a two folios space, let us discover how $r(\varphi)$, given by equations (21, 22), shows the precession of the image in space of the geodesic, in spacetime.

All this figures, [23] represent the function $r(\varphi) = L/(-1+2(Sin-Jacobi-Amp(\varphi/2K, A))^2)$ for $L/GM = 16$, (with $c = 1$), in polar coordinates. The point C ($r = 0$), center of the polar representation (the angle coordinate is φ), is far outside of the figures. The distance CO corresponds to the minimum length (in absolute value) of the radial coordinate r (in this example $r_{min} = 16$).

Figure 2a, for $0 \leq \varphi \leq \pi$, displays two half branch (OA) and (OB) of two different, twisted, seudo-hyperbolas On figure 2b, for $0 \leq \varphi \leq 2\pi$, we see that a twisted and rotated half branch (OC) is added to (OA) and that a twisted and rotated half branch (OD) is added to (OB).

On the figure 2c, for $0 \leq \varphi \leq 4\pi$, another similar set (A'OC') and (B'OD') of two, twisted, branches of different pseudo-hyperbolas, positively rotated, relatively to (AOC) and (BOD), appears.

This shows that the image in space of the precession phenomenology in spacetime also exhibits a precession phenomenology. Obviously, at each addition of 2π to φ, $\varphi + 2\pi(n+1)$, a new twisted, rotated, set of branches of pseudo hyperbola will appear (see figure 2d).

[19]Figure 1a,1b are drawn by using Maxima 14.12.1. If we use $r = 1/u$ instead of u ,we get:($-l \leq r$ for $0 \leq \theta(r) \leq \pi/4$ and $r \geq l$ for $\pi/4 \leq \theta(r) \leq \pi/2$), which is less convenient as it is not a continuous domain.

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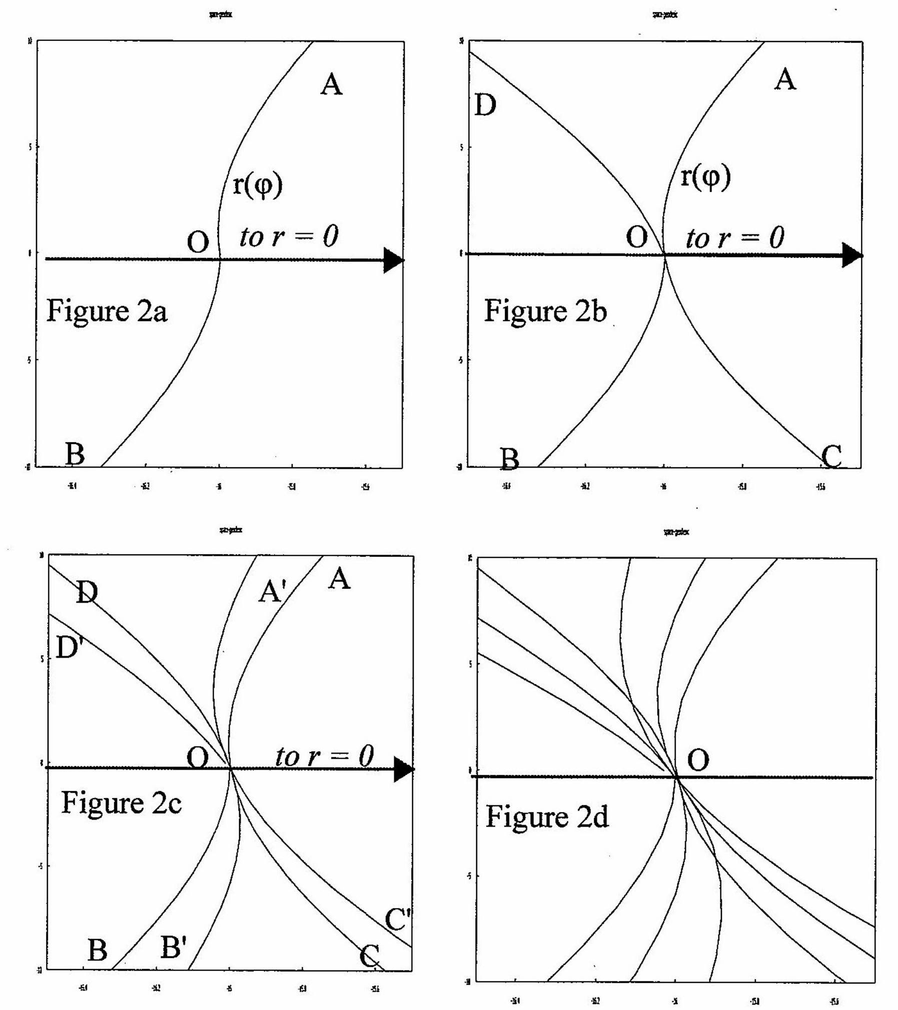
[21]The metric of this 3D space is not singular for $r = 0$

[22]Let us notice that this is also true in spacetime. In [?],pages 100-105, this is quite well explicated.

[23]Drawn with wxMaxima 14.12.1. For $L/GM = 16$, $c = 1$, $2K = 1.886$, $A^2 = 0.222$. See definition of K and A^2 in equation (15).


A
r(φ)
O
to r = 0
Figure 2a
B
D
A
r(φ)
O
to r = 0
Figure 2b
B
C
D
A'
A
D'
O
to r = 0
Figure 2c
C'
B
B'
C
O
Figure 2d

D Acknowledgments

I am grateful to the professor Jean Jacques Szczeciniarz [24] for his advices and his support, for this article. I thank Denis Mercier [25] for his kind support on some mathematical issues. I thank Olivier Darrigol [26] for providing references to analytic solutions and Marc Lachièze-Rey [27] for his comments and suggestions.

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