Precession of geodesics in space section of Schwarzschild's spacetime

Fric Jacques*

June 14, 2024

Summary The precession, resulting from the 4D spacetime curvature, of spacetime geodesics in Schwarzschild's spacetime, is well-known topic.[1]. The geodesic curve in this spacetime is fully defined by a $r(\varphi)$ function on a curve defined in 2D space. In this article, after a review of spacetime geodesics and of their space sections, we will derive the precession of a geodesic in the space section of the Schwarzschild's spacetime. Even though a space section of a spacetime has no physical character, it might enlightens the understanding of the geometry of the Schwarzschild's spacetime. This precession in this space section will only depends on the ratio M/l, is defined by an original infinite polynomial of even powers of M/l, this providing a real precession, even for imaginary values of l.[2]

1 Spacetime and space geodesics in Schwarzschild's metric

For defining the precession of the perihelion of a geodesic in Schwarzschild's 4D spacetime, a function $r(\varphi)$, depending on parameters M (mass of the central body) and on the parameters of the geodesics :$L = r^2 d\varphi/d\tau$ (angular momentum) and $E = (1-2GM/r)dt/d\tau$ (energy), is only needed. The geodesic is a curve in spacetime (see figure 1 in annex 1), where per the spherical symmetry, the space section of the geodesic is a curve included in a 2D plane ($\theta = constant$, usually one set $\theta = \pi/2$)[3]. In the three dimensional space section of Schwarzschild's spacetime where $d\sigma^2$ is the metric line element, we can also to define the precession of a geodesic in this space section by a $r(\varphi)$ function, with an angular momentum defined by $l = r^2 d\varphi/d\sigma$. An interesting feature is that, even though the related **spacelike** angular momentum l is an **imaginary** parameter, in the usual convention where L (timelike parameter) is real, the space precession, which is a physical phenomenon, is surprisingly real! [4]

Original Schwarzschild's metric is recalled in equation (1) below.

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 \ + \sin^2\theta d\varphi^2) = -(1 - \frac{2GM}{r})dt^2 + d\sigma^2 \tag{1}$$

Where,

$$d\sigma^2 = +\frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 \ + \sin^2\theta d\varphi^2) = \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\varphi^2, \text{ for } \theta = \frac{\pi}{2} \tag{2}$$

In the last part of the equation , we set $\theta = \pi/2$, which is allowed, per the spherical symmetry, without loss of generality.

*jacques.fric@etu.univ-paris-diderot.fr. Paris-Diderot University, Laboratory SPHERE.

[1] For analytic solution solving the spacetime equation, see for instance [9], chapter 19.

[2] M is the mass of the central body and l is a generic notation for the angular momentum in spacetime or in space.

[3] This curve which is the projection of the spacetime geodesic on this plane is not a geodesic in space, of affine parameter σ used in the line element $d\sigma^2$ of the space metric, see annex 1.

[4] In 4D spacetime, the affine (dynamic) parameter on a timelike geodesic is the proper time, a timelike parameter. In space the affine parameter on the spacelike geodesic, is a spacelike parameter therefore the dimension of $L = r^2 d\varphi/d\tau$ will be a square length divided by a time while that of $l = r^2 d\varphi/d\sigma$ will be a length.

2 Analytic method for the precession of planets in Schwarzschild's spacetime

2.1 Interest of geodesics in a space section of spacetime

General relativity is a geometrical theory of the gravitation in spacetime. Physical geodesics are timelike or null geodesics. [5] Unlike geodesics in spacetime which do not depend on the coordinates, geodesics in a space section of spacetime [6] depend on the coordinates. So, what would be the interest of such geodesics? We may select a space section in spacetime, but we know that this depends on the selected coordinates, and that this space section (at constant time in the coordinates attached at this frame), is arbitrary [7]. This interest is motivated by an original proposal of Painlevé [7] describing the Schwarzschild's spacetime geodesic as the geodesic of the Schwarzchild space section multiplied by a conformal factor. This shows that the conformal structure of the space section and that of the spacetime are the same. This is a quite surprising property. See annex 1 for some details.

But, even though a space section is arbitrary, the space section in the coordinates used in the Schwarzschild frame as described in equation (1) may be interesting because on the one hand the time is "'orthogonal"' to the space section [8] and on the other hand the form of the metric does not depend on time. Therefore, when one defines the spacetime geodesic by using the Schwarzschild coordinates, geodesics in its space section would provide a complementary information for describing the underlying geometrical space structure of a space section which is more complex than it appears at a first look, see [10], figure 1.

2.2 The geodesic equation in space

We will use the well-known method for getting a general solution for the geodesic equation in space and then, we will use it for solving the problem of the precession of geodesics in space. Dividing second part of equation (2) by $d\sigma^2$ yields:

$$1 = \frac{dr^2}{d\sigma^2(1-\frac{2GM}{r})} + r^2\frac{d\varphi^2}{d\sigma^2} \Rightarrow \frac{dr^2}{d\sigma^2} = (1-\frac{2GM}{r})(1-r^2\frac{d\varphi^2}{d\sigma^2}) \tag{3}$$

$$\frac{dr^2}{d\sigma^2} = (1-\frac{2GM}{r})(1-\frac{l^2}{r^2}) \tag{4}$$

Equation (4), valid only on a geodesic, is equation (3) with $l = r^2 d\varphi/d\sigma$ which is the conserved angular momentum, on the spatial geodesic.[9]

By multiplying equation (4) by $(d\sigma/d\varphi)^2 = r^4/l^2$, we get:

$$\frac{dr^2}{d\varphi^2} = (1-\frac{2GM}{r})(\frac{r^4}{l^2}-r^2) \Rightarrow d\varphi = \frac{\pm dr}{\sqrt{-r^2(1-\frac{2GM}{r})(1-\frac{r^2}{l^2})}} \tag{5}$$

Let us set:

$$u = \frac{1}{r} \Rightarrow r = \frac{1}{u} \Rightarrow dr = -\frac{du}{u^2} \tag{6}$$

By inserting it, in equation (5), we get :

$$\frac{du^2}{u^4 d\varphi^2} = (1-2GMu)(\frac{1}{u^4})(\frac{1}{l^2}-u^2) \Rightarrow d\varphi = \frac{\pm du}{\sqrt{(1-2GMu)(\frac{1}{l^2}-u^2)}} \tag{7}$$

By defining an angle θ, a parameter A^2 and a constant K, such as:

$$\theta = \arcsin\sqrt{\frac{1+lu}{2}} \Rightarrow \sin^2\theta = \frac{1+lu}{2}, A^2 = \frac{4GM}{2GM+l}, K = \sqrt{\frac{l}{l+2GM}} \tag{8}$$

[5] One can also define spacelike geodesics in spacetime but there are not considered to be physical.

[6] In a space section of the Schwarzschild 's spacetime, the coordinate t is constant.

[7] We will select a space section in the Schwarzschild's coordinates, but a space section in the Painlevé's coordinates, describing the same spacetime would return an Euclidean space section whose geodesics are straight lines!

[8] This means that the four basis vectors, associated to the coordinates are orthogonal according to the definition of orthogonality in relativity.

[9] This "'constant of motion"' l exists as the metric $d\sigma^2$ does not depend on φ. But unlike the angular momentum L in spacetime which is physical, the space angular momentum l will depend on the choice of the space section.

Equation (7) can be written:[10]

$$\frac{d\varphi}{2} = K\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} \Rightarrow \frac{\varphi(\psi,A^2)}{2} = K\int_{\theta=0}^{\theta=\psi}\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} = K Elliptic_F(\psi,A^2) \tag{9}$$

Inserting the values of θ, K and A^2 defined in equation (8) yields:

$$\frac{\varphi}{2} = \sqrt{\frac{l}{l+2GM}} Elliptic_F[\arcsin(\sqrt{\frac{1}{2}(1+lu)}), \frac{4GM}{l+2GM}] \tag{10}$$

$Elliptic_F(\psi,A^2)$ or $F(\psi,A^2)$ in a short notation is the integral described in equation (9). This integral, called elliptic integral of the first kind, includes an argument A^2 called the parameter, A is called the modulus.[11] The parameter $\psi = \theta(u)$, called the amplitude, is, as shown in equation (9), the upper limit of integration of the angle θ defined in equation (8).

In equation (10), in the $Elliptic_F$ integral, the constraint on the parameter $\theta = \arcsin\sqrt{x}$ implies $0 \le x \le 1 \to -1/l \le u \le 1/l$ and the constraint $A^2 \le 1$, implies that $2GM \le l$ We know that the Schwarzschild's solution is not the maximally extended solution for this spacetime as it describes only two spacetime regions, denoted I and II, of the four spacetime regions, denoted I, II, III, IV, described, for instance, by the Kruskal's solution. The negative value of u and r, as $r = 1/u$, should be associated to the space sections of regions III and IV of the Kruskal solution. This shows that, even though the Schwarzchild's solution does not describes all the spacetime regions, its space section solution describes all space sections of the spacetime. This is possible as there is no singularity for $r = 0$ in the space metric and because the singularity at $r = 2GM$ is not physical. As, per the definition of the $Elliptic_F$ integral, $A^2 \le 1 \to l \ge 2GM$, this implies, in addition, that in this problem, only space sections of regions I and IV (outside of this horizon) are involved.

Returning to $r = 1/u$, we get:

$$\frac{\varphi}{2} = \sqrt{\frac{l}{l+2GM}} Elliptic_F[\arcsin(\sqrt{\frac{1}{2}(1+\frac{l}{r})}), \frac{4GM}{l+2GM}] \tag{11}$$

2.3 Elliptic-K

$$K(A^2) = Elliptic_K(A^2) = Elliptic_F(\frac{\pi}{2}, A^2) = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} \tag{12}$$

$Elliptic_K(A^2)$, also called $K(A^2)$ in a short notation, is a special case of $Elliptic_F$, where the upper limit ψ is equal to $\pi/2$. It is the complete elliptic integral of the first kind of Legendre and therefore has only one parameter (A^2). There exists an analytic definition of the integral $Elliptic_K(A^2)$ by an infinite polynomial of powers of A^2. This is this polynomial definition, given by equation (19), that we will use further, for calculating the precession.

With the values of K and A^2, defined in equation (8), by using the definition of $Elliptic_K(A^2)$, the equation (10) for $\psi = \pi/2$, can be written:

$$\frac{\varphi(\psi=\pi/2,A^2)}{2} = \sqrt{\frac{l}{l+2GM}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} = \sqrt{\frac{l}{l+2GM}} Elliptic_K(\frac{4GM}{l+2GM}) \tag{13}$$

Equation(13) that we will use for computing the precession, defines $\varphi/2$ and not φ therefore, we will have to multiply it by two for getting the result. Equation (13) shows that the solution in space only depends on the parameters M and l.

2.4 Jacobi-Amp-function

The $Elliptic_F$ integral defined in equation (10) $\varphi/2K = F(\psi,A^2)$ has an inverse function called Jacobi-Amplitude function noted $am(\varphi/2K, A^2)$, such that $\psi = am(\varphi/2K, A^2)$. So per the definition of $\sin(\psi)$,

[10] By using the definition of θ, A^2 and K in equation (8), annex 2 shows that it is straightforward to verify that $du^2/[(1-2GMu)(-u^2+1/l^2)] = (2Kd\theta)^2/(1-A^2\sin^2\theta)$ whose integral is an elliptic integral of first kind.The binary operator $\pm$, in the last term of equation (7), is related to the orientation of the geodesic as there are two possible orientations. For the integration, we can select one direction, without loss of generality, that we will associate to the sign $+$.

[11] There are several formal notations, this being quite confusing. For instance, it is denoted $Elliptic_F(\psi,A)$ in WolframMathWorld but, in both notations, it is A^2 which is used in the computation of the integral. It is just two notations for the same object. This remark will also apply to the $Elliptic_K$ integral and Jacobi-Amp function, that we will use further.

where ψ is the upper limit of θ, and K in equation (8) we get [12]:

$$\sin^2\psi = \frac{1+lu}{2} = sn^2(\frac{\varphi}{2K}, A^2) \Rightarrow u = \frac{1}{l}(-1+2sn^2(\frac{\varphi}{2K}, A^2)) \quad (14)$$

This gives the function $u(\varphi)$. The function $r(\varphi)$ can be deduced by using the relation $r = 1/u$.

2.5 These elliptic integrals define the precession

$Ellipic_F, Elliptic_K$ and their inverse integrals exhibit two angles ψ and φ. Equation (13) shows that when θ varies from *0* to $\psi = \pi/2$, $\varphi/2K$ varies from *0* to $Elliptic_K(A^2)$. So for a half-pseudo-orbit where θ varies from *0* to π [13]:

$$\Delta(\frac{\varphi}{2}) = 2K Elliptic_K(A^2) - \pi \quad (15)$$

And equation (16) below will be the equation to be used for solving the problem for n full orbits.

$$\Delta\varphi = 4n(2K Elliptic_K(A^2) - \pi) \quad (16)$$

2.6 Class of equivalence

The equation (16) only depends on parameters K and A^2. This equation will provide an exact solution to the problem, if we know these parameters on the spacelike geodesic. By posing $2GM/c^2l = k^2$, parameters K and A^2, defined in equation (8), can be written:

$$K = (1+\frac{2GM}{c^2l})^{-1/2} = (1+k^2)^{-1/2}, A^2 = (\frac{2GM}{c^2l})(\frac{2}{1+\frac{2GM}{c^2l}}) = (2k^2/(1+k^2)) \quad (17)$$

Therefore, equation (16) will only depend on the dimensionless parameter k^2. We will expect a solution as a function of k^2.

This parameter k^2 defines a class of spatial solutions.

3 Solution for the precession in space

3.1 General solution

By inserting the definition of A^2 and K, given in equation (17), in equation (13), giving the formal general solution, for $\psi = \pi/2$, we get:

$$\frac{\varphi}{2} = \sqrt{\frac{1}{1+k^2}} Elliptic_K(\frac{2k^2}{k^2+1}) \quad (18)$$

The integral $Elliptick_K(k)$ can be represented by an infinite polynomial: [14]

$$Elliptic_K(\frac{2k^2}{k^2+1}) = \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n!)}{2^{2n}(n!)^2}]^2(\frac{2k^2}{k^2+1})^n \equiv \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(\frac{2k^2}{k^2+1})^n \quad (19)$$

where n!! denotes the semi-factorial. By using equation(19), equation (18) becomes:

$$\frac{\varphi}{2} = \sqrt{\frac{1}{1+k^2}}(\frac{\pi}{2})\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(\frac{2k^2}{1+k^2})^n = \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(2k^2)^n(1+k^2)^{-\frac{1}{2}(2n+1)} \quad (20)$$

We replaced $-(n+1/2)$ by $-(1/2)(2n+1)$ which will be more convenient. Let us recall that $(1+k^2)^{-\frac{1}{2}(2n+1)}$ can also be developed in an infinite polynomial, as defined below:

$$(1+k^2)^\alpha = 1 + \sum_{j=1}^{j=\infty}\frac{(\alpha)(\alpha-1)..(\alpha-j+1)}{j!}(k^2)^j \quad (21)$$

[12] In Jacobi elliptic functions, $sn(\varphi/2K, A^2) = sin(am(\varphi/2K, A^2))$, see WolframMathWorld, Jacobi elliptic functions.

[13] A half-orbit defines the dynamic as we assume the symmetry of the orbit for the precession.

[14] $http://mathworld.wolfram.com/CompleteEllipticIntegraloftheFirstKind.html$, equation (2). In terms of the Gauss hypergeometric function, $Elliptic_K = (\pi/2)_2F_1(1/2, 1/2, 1, 2k^2/(1+k^2))$. Let us recall that the Gauss hypergeometric function ${}_2F_1(a,b,c;z)$ is a solution of the second order homogeneous differential equation $z(1-z)d^2y/dz^2 + [c-(a+b+1)z]dy/dz - aby = 0$.

For $-1 < k^2 < 1$ and where α is a real number, with $\alpha = -(1/2)(2n+1)$ in our problem.

By using these formulas, equation (20) becomes: [15]

$$\frac{\varphi}{2} = \frac{\pi}{2}\sum_{n=0}^{n=\infty}(\frac{(2n-1)!!}{(2n)!!})^2(2k^2)^n(1+\sum_{j=1}^{j=\infty}\frac{(2n+1)(2n+3)..(2n+1+2(j-1))}{(-2^j)j!}(k^2)^j) \tag{22}$$

The product of these two infinite polynomials is an infinite polynomial. For defining this polynomial we have to calculate each coefficient B_n of $(k^2)^n$ which, in the infinite polynomial defined in equation (22), will be the sum of the product of coefficients of the terms$(k^2)^i$ of the first polynomial with the coefficients of the terms $(k^2)^{n-i}$ of the second polynomial.

The result of this operation will define an infinite polynomial $P(k^2)$

$$P(k^2) = \frac{\pi}{2}\sum_{n=0}^{n=\infty} B_n(k^2)^n \tag{23}$$

Where in B_n, as the first factor of the sum is related to $(k^2)^i$, the second factor should be related to $(k^2)^{n-i}$, therefore in equation (22), we must set $n = i$, $j = (n-i)$. This yields: $(2i+1)(2i+3)..(2i+1+(2(n-i-1))) = (2i+1)(2i+3)..(2n-1)) = (2n-1)!!/(2i-1)!!$. Therefore B_n can be written:

$$B_n(k^2)^n = \sum_{i=0}^{i=n}(\frac{(2i-1)!!}{2i!!})^2(2^i(k^2)^i)(\frac{(2n-1)!!}{(2i-1)!!(n-i)!(-2^{n-i})})(k^2)^{n-i} \tag{24}$$

By simplifying by $(2i-1)!!$ and using $(2i-1)!! = 2i!/(i!2^i)$, $(2i)!! = i!2^i$, $2i!/i!^2 = 2i!/(i!(2i-i)!) = \binom{2i}{i}$, $n!/i!(n-i)! = \binom{n}{i}$, this equation yields:

$$B_n = \frac{2n!}{n!^2}(2^{-2n})\sum_{i=0}^{i=n}\frac{(-1)^{n-i}}{2^i}\binom{n}{i}\binom{2i}{i} = \frac{2n!}{n!^2}(-1^n)(2^{-2n})\sum_{i=0}^{i=n}(-\frac{1}{2})^i\binom{n}{i}\binom{2i}{i} \tag{25}$$

3.2 The polynomial includes only even powers of (k^2)

We will demonstrate that equation (25) giving B_n is a hypergeometric series. Such series is defined by using the Gauss hypergeometric function ${}_2F_1(a;b;c,d)$. [16]

Let us set:

$$A(n) = \frac{2n!}{n!^2}(-1^n)(2^{-2n}) \tag{26}$$

Per the formal definition of the Gauss hypergeometric function:

$${}_2F_1(a;b;c,d) = \sum_{i=0}^{i=n}\frac{(a)_i(b)_i}{(c)_i}\frac{(d)^i}{i!} \tag{27}$$

Where the notation $(a)_i = a(a+1)..(a+i-1)$ is the Pochhammer symbol. If we set $a = -n$, $b = 1/2$, $c = 1$, $d = 2$, we get:

$(a)_i = (-n)(-n+1)..(-n+i-1) = (-1)^i(n)(n-1)..(n-i+1) = (-1)^i n!/(n-i)!$, $(b)_i = (1/2)(3/2)...(2i-1)/2 = (1/2)^i(2i-1)!! = (1/2)^i 2i!/(2^i i!)$, $(c)_i = (1)(2)...(i) = i!$, $d^i = 2^i$

Inserting these values in equation (27) yields:

$${}_2F_1(-n;1/2;1,2) = \sum_{i=0}^{i=n}(\frac{(-1)^i(n!)}{(n-i)!})((\frac{1}{2})^i\frac{2i!}{2^i i!})(\frac{1}{i!})(\frac{2^i}{i!}) = \sum_{i=0}^{i=n}(-\frac{1}{2})^i\binom{n}{i}\binom{2i}{i}) = \frac{B_n}{A(n)} \tag{28}$$

In the second sum of the equation above we used the relations: $n!/((n-i)!i!) = \binom{n}{i}$ and $2i!/(i!i!) = \binom{2i}{i}$.

This is the result that we expected!

For demonstrating that all terms $B_{2m+1}(k^2)^{2m+1}$ vanish, we need to use equation (16) of mathworld.wolfram-HypergeometricFunction, which gives an integral defining the hypergeometric function.

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[15] In the second sum, we will separate the factor $1/2$ and the sign $-$ from the formula and will gather them in the factor $1/(-2^j)$ for simplifying the calculation.

[16] See mathworld.wolfram-hypergeometricFunction, equation (8) for the form of the series generated by such function, and the Pochhammer notation symbol $(a)_n$.

$$ {}_2F_1(a;b;c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a} du. \tag{29} $$

By calculating this integral with our parameters ($a=-n, b=1/2, c=1, z=2$), we get:

$$ \int_0^1 u^{-1/2}(1-u)^{-1/2}(1-2u)^n du = \int_0^{0.5} u^{-1/2}(1-u)^{-1/2}(1-2u)^n du + \int_{0.5}^1 u^{-1/2}(1-u)^{-1/2}(1-2u)^n du \tag{30} $$

Let us define two values of u, $(0 \leq u \leq 1)$, u_1 and u_2 such that $u_1 = 1/2 + a$ and $u_2 = 1/2 - a$, where $a \leq (1/2)$. We get $u_1^{-1/2}(1-u_1)^{-1/2} = [(1/2+a)(1/2-a)]^{-1/2} = u_2^{-1/2}(1-u_2)^{-1/2} = [(1/2-a)(1/2+a)]^{-1/2}$ (symmetry around 1/2).

In $(1-2u)$, for $u = u_1$ we get $-2a$ and for $u = u_2$, we get $2a$. This, raised to power n, will give $(u_1)^n = (-2a)^n$ and $(u_2)^n = (2a)^n$ which are equal when n is even and are opposite when n is odd.

Therefore, as exhibited by equation(30), where the integral is split in two parts (from 0 to 1/2 and from 1/2 to 1), when n is odd ($n = 2m+1$), the two parts are opposite, the integral vanishes and when n is even ($n = 2m$) the two parts are equal, this integral does not vanish.

Therefore:

$$ P(k^2) = \sum_{n=0}^{n=\infty} B_{2n}(k^2)^{2n} \tag{31} $$

includes only even powers of k^2.

$$ \frac{\varphi}{2} = K.Elliptic_K(A^2) = P(k^2) = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(k^4)^n = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(\frac{(2GM)^2}{(cL)^2})^n \tag{32} $$

The form of this polynomial, describing the precession, shows that, even for an imaginary value of the angular momentum, a real number for φ and therefore for the precession is expected! This is important because, per the form of the metric described in equations (1) and (2), if we assume that the angular momentum, on the timelike geodesic in spacetime, is real, that on the spacelike geodesic in space will be imaginary. Per this property, both precession in spacetime and space will be real.

A numerical value of this polynomial up to $n = 5$ is provided in equation (33).

3.3 Numerical value of the polynomial

The polynomial $P(k^2)$ is given below up to $n = 10$. [17]

$$ P(k^2) = \frac{\pi}{2}(1 + \frac{3}{16}(k^4) + \frac{105}{1024}(k^8) + \frac{1155}{16384}(k^{12}) + \frac{225225}{4194304}(k^{16}) + \frac{2909907}{67108864}(k^{20})) \tag{33} $$

4 Review of possible relations between spacelike, timelike and null geodesics

4.1 Spacetime timelike geodesic versus spacelike geodesic in space section of spacetime

One may wonder whether the precession in a space section of the Schwarzschild's spacetime is a part of the precession in spacetime and, in case, how the curvature of time [18] may fill the gap. But this is likely not the case as the geodesic in space is generally [19] not the projection of the timelike geodesic in the space section of Schwarzschild's spacetime (see annex 1). The relation between geodesics in spacetime and in the space section of the Schwarzschild's spacetime should be more subtle.

The special case of circular orbits in spacetime, which has an analytic solution [20], as described in [1], pages 208-212 equations (5.71)- (5.72) for instance, even though it is an exception, may illustrate their relation. In

[17] The coefficients B_n are computed by using mathematica line of command : $Bn = FullSimplify[((2n)!/(n!)^2)(2^{-2n}((-1)^n)Sum[((-1/2)^j)Binomial[n,j]Binomial[2j,j],(j,0,n)]]$, for $n = 2,4,6,8,10$.

[18] The dynamic parameter ruling the dynamic of the spacetime equations is the proper time which, as we know, is different of the coordinate time.

[19] The case of circular orbits is an exception as we will show.

[20] We get the solution by solving the equation $d[V(r)]/dr = 0$ for getting the extrema of *V(r)*. This yields a second degree equation with two roots.

Newtonian theory the radius of a circular orbit is $r_c = L^2/GM$. In relativity, in weak field for $(L^2/GM >> 1)$, we get two values for the radius corresponding to the extrema of *V(r)*, for a given value of the relativistic angular momentum. One is $r_{c1} \approx L^2/GM$ for a minimum of the potential *V(r)* (stable orbit for an orbiting body) and the other $r_{c2} \approx 3GM$ for the maximum of *V(r)* (unstable orbit for an orbiting body). Both are physical orbits in a sense that a physical body can move on these orbits. Usually, we are interested in stable orbits in spacetime [21].

In the space section, by using the same method for getting orbits at constant value of the coordinate r, but by using the radial potential $W(r)$, we get also two values for the extrema of the radius which have the same value that those in spacetime but of opposite kind (to a minimum in *V(r)* corresponds a maximum in *W(r)* and vice versa) as the derivative of second order of *W(r)* shows. This was obvious according to the relation $W(r) = -V(r)$.

Usually spacelike geodesics are considered as unphysical, as no physical known body can "'move"' on such geodesics. If we assume that the angular momentum L in spacetime is real, the angular momentum on the geodesic at constant radius r_c is represented by an imaginary number as $l = iL$. in the space section. But, because in *V(r)* and *W(r)* are included only L^2 and $-L^2$ terms, which are real number the geodesics are also real and circular. Here the attribute stable for the geodesic associated to the minimum of the potential *W(r)* or unstable for that associated to the maximum of *W(r)* makes no sense. These geodesics are well defined curves in the space section of the Schwarzschild's spacetime.

The *V(r)* and *W(r)* potential curves are represented respectively on figure 2a and figure 2b of annex 3.

Obviously, in the case of a circular geodesic this does not involve a precession of the perihelion (or aphelion) as there are no perihelion or aphelion and therefore no change in the spatial and time curvature along the geodesic. The spacetime precession of the perihelion of a non circular geodesic results from the variation of the spacetime curvature on the geodesic.

4.2 Spacetime timelike geodesic versus spacelike geodesic in space section of spacetime, in weak field

Comparing a space geodesic with a spacetime timelike geodesic in Schwarzschild spacetime is not straightforward because, if the parameter of mass M of the central body may be common to both geodesics, we already stated that the angular momenta L in spacetime and the angular momentum l in space have different dimensional definition. Moreover, in the spacetime timelike geodesic, associated to the additional coordinate time by the relation $E = (1 - 2GM/r \times c^2)dt/d\tau$, the geodesic depends on an additional parameter which is the energy per unit mass. [22]

But, even though this is not true in strong field, it is worth to notice that, in weak field, the precession of the geodesic in the space section is half of that of the precession in spacetime, even though if this applies on different geodesics.

In annex 3, by using the radial potential $V(r)$ in spacetime, as described in [1], equation (5.66), page 209, that we extend to a radial potential $W(r)$ in space, we show that $V(r) = -W(r)$ if $c^2 l^2 = -L^2$.

4.3 null geodesics in spacetime

Moreover, the comparison in weak field of the space geodesic and of the null geodesic for a same mass M and a same angular momentum l, which has the same dimension in both geodesics, exhibits an interesting property: The precession given by the elliptic integral defining the geodesic equation in space (equation (11), is half of the geodesic deflection of light (null geodesic).

5 Conclusion

In this paper, we focused our analysis on the space geodesic which does not look to be the most important in the theory of general relativity which is a spacetime theory. But as the most important information that we

[21] Let us emphasize that describing a geodesic in general relativity by the motion of a fictitious unitary mass, on it, is just a way to illustrate the phenomenology. In fact the geodesic exists, without any test body on it, as a fully defined curve in spacetime therefore, having both space and time attributes. Therefore, stable or unstable is not really an attribute of the geodesic itself which is perfectly defined. It is an attribute of the spacetime around it. Strictly speaking, the insertion of a physical body on this curve invokes a local coupling between this body and the spacetime described by the general relativity. In general, this is not explicitly claimed because we assume that this body is small enough, in size and in mass, for not disturbing, at least at first order, the phenomenology. Nevertheless this body may be subject to small perturbations, whose effects on the phenomenology may differ according to the kind of geodesic.

[22] The energy is also described by the energy of a "'test particle of unitary mass"', but, strictly speaking, its definition shows that this energy is an attribute of the geodesic in spacetime

get in physics about some physical or geometrical objects does not reside in the objects themselves but in their relations, this complementary analysis which provides a set additional relations between objects may induce a better understanding of the underlying physics described by the theory.

A Annex 1: Geodesic in spacetime and its projection in space

On the figure 1, we represented the geodesic in spacetime defined by the points $m(_{t=i})$ for $i = 0, 1, 2, 3$, of affine parameter τ and its projection in space defined by the points $m(_{t=0})$ and $pm(_{t=i})$ for $i = 1, 2, 3$, of affine parameter σ, where i is the index of the time slice (a plane) on the time coordinate represented on a vertical axis. We represented 4 planes corresponding to constant values of the coordinate time: $t = 0, t = 1, t = 2, t = 3$. We suppressed one dimension (that corresponding to θ which is set to $\pi/2$). The worldline followed by the observer at the center of the Solar frame, defined by the points $O(t = 0)$ up to $O(t = 3)$, is the vertical time axis. We see that, for $t = 0$, the observer is in $O(t = 0)$ on its worldline and we see that the test mass m is in $m(t = 0)$, of space coordinates $r = r_0$ and $\varphi = 0$, in the same time slice. At $t = 1$, the observer is in $O(t = 1)$ on its worldline and he measures the angle $\varphi = \varphi_1$,corresponding to the current position of the test mass in $m(t = 1)$ at a distance $r = r_1$, on the geodesic in spacetime, in the time slice $t = 1$ and so on for the other time slices $t = 2, t = 3$. For the measurement of angles this is equivalent to a measurement by a fictitious observer, remaining in the $t = 0$ plane, of the position of the projection on the plane $t = 0$ of the test mass on its spacetime geodesic. We also represented $d\tau$ on the geodesic and $d\sigma$, its projection on time slice $t = 0$. The angular momentum in spacetime $L = r^2 d\varphi/d\tau$, where τ is the affine parameter, is constant on the spacetime geodesic. The coordinates r and φ which are the space coordinates in a time slice are also those on the projection of the spacetime geodesic on the time slice *0*.

An interesting relation was provided by P. Painlevé, [2],[3], [4], [8] in [7] page 1141 in postulate VI, for a geodesic equation in spacetime. His relation was relying on a prior development of a geometrical formalism of Newtonian mechanics [6], page 876 [5]. [23]

Following and correcting his proposal, we can write:

$$ds^2 = (\frac{U}{E^2 - U})(d\sigma)^2 = (\frac{1 - \frac{2GM}{r}}{E^2 - (1 - \frac{2GM}{r})})(\frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\varphi^2) \Rightarrow -E^2 + (\frac{dr}{ds})^2 + (1 - \frac{2GM}{r})(\frac{L^2}{r^2} + \epsilon) = 0 \quad (35)$$

A first comment is that the line element of the geodesic [24] curved spacetime metric (ds^2) can be written as a product of the line element of the curved space section of the Schwarzschild's metric ($d\sigma^2$), the curvature of which is depending only on the gravitational potential ($U = 2GM/r$) by a "‘conformal factor"’ (($U/(E^2 - U)$), depending also on the gravitational potential U and, in addition, of the energy E, associated to the time in general relativity, of a test particle of unitary mass. The time is therefore inserted in the spacetime metric by the energy, constant on a geodesic.[25] By setting $c = 1$, we can write $ds = d\tau$ in equation (35) we can deduce from this equation that $d\tau/d\sigma = \sqrt{U/(E^2 - U}$. Per the definition of $L = r^2 d\varphi/d\tau$ and $l = r^2 d\varphi/d\sigma$ therefore if we apply this relation on the same curve defined by $r(\varphi)$ which is the projection on the plane defined by $t = 0$ of the spacetime geodesic, we can deduce $l/L = d\tau/d\sigma = \sqrt{U/(E^2 - U}$ where $U = 1 - 2GM/r$. As L is a constant (on this curve), this relation which depends on the coordinate r which varies on this curve shows that l is not a constant (on this curve). Therefore, except for a circular orbit [26], this curve is not a geodesic of affine parameter σ.

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[23] In fact, in [7] page 1141 , Painlevé wrote:

$$ds^2 = (U + h)(d\sigma)^2 \quad (34)$$

where $U = r/(r - 2GM)$ and where h is constant associated to the conserved energy. It is straightforward to verify that the conformal factor is not correct, we corrected his error. One can verify by developing equation (35) and dividing it by ds^2, with $L = r^2 d\varphi/ds$ on the geodesic, that this yields the spacetime geodesic equation (where $\epsilon = 1$ for a timelike geodesic), see for instance [1] equation (5.64) page 208.

[24] This above equation is valid only for the metric on a geodesic

[25] The above equation shows that, per the conformal relation between the space metric and the spacetime geodesic metric, their conformal structure is the same but this does not constrain the geodesic structure which is not invariant by conformal transformations.

[26] Where $U = 1 - 2GM/r_c$ is constant as r_c is constant. Per the definition of $U = 1 - 2GM/r$ and $E^2 = dr^2/d\tau^2 + 1 - 2GM/r + L^2/r^2 - 2GML^2/r^3$ where $dr^2/d\tau^2 = 0$ in this case, we get $d\tau^2/d\sigma^2 = U/(E^2 - U) = r^2/L^2$. The conformal factor is constant. Therefore as $d\tau/d\sigma = \pm r/L$, σ is an affine parameter of the spatial curve which is the projection of the spacetime geodesic on the space section of the Schwarzschild's spacetime

C:/Users/jacques/Documents/precession-3/spacetime-geodesic.jp

B Annex 2: Form of the geodesic equation

We have to check that:

$$\frac{d\varphi^2}{4} = \frac{K^2 d\theta^2}{1 - A^2 \sin^2\theta} \text{ for } \sin\theta = \sqrt{\frac{1+lu}{2}}, A^2 = \frac{4GM}{2GM+l}, K = \sqrt{\frac{l}{l+2GM}} \tag{36}$$

is equivalent to:

$$\frac{l^2 du^2}{(1-2GMu)(1-u^2l^2)} \tag{37}$$

By taking the derivative of $sin(\theta)$, defined in eq. (36), we get:

$$d\theta^2 \cos^2\theta = \frac{l^2 du^2}{8(1+lu)} \Rightarrow d\theta^2(1-\sin^2\theta) = \frac{l^2 du^2}{8(1+lu)} \tag{38}$$

$$\frac{d\theta^2}{2}(1-lu) = \frac{l^2 du^2}{8(1+lu)} \Rightarrow d\theta^2 = \frac{l^2 du^2}{4(1-l^2u^2)} \tag{39}$$

Therefore

$$d\varphi^2 = \frac{4K^2 d\theta^2}{1 - A^2\sin^2\theta} = \frac{4l}{l+2GM} \frac{l^2(2GM+l)du^2}{4(1-l^2u^2)(2GM+l-2GM(1+lu))} \tag{40}$$

By simplifying, this equation we get:

$$d\varphi^2 = \frac{l^2 du^2}{(1-2GMu)(1-u^2l^2)} = \frac{du^2}{(1-2GMu)(\frac{1}{l^2}-u^2)} \Rightarrow d\varphi = \frac{du}{\sqrt{(1-2GMu)(\frac{1}{l^2}-u^2)}} \tag{41}$$

which is the equation (7), as expected.

C Annexe 3: Interest of a weak field solution, in this method

In weak field, we would like to use the value of the Newtonian angular momentum C. For doing that we need to know the relation between L and C as their definitions are different.

C.1 Relation between relativistic angular momentum and Newtonian angular momentum in weak field

In spacetime, the geodesic equation is equation (5.64) page 208 in [1] :
$-E^2 + (dr/d\tau)^2 + (1-2GM/r)(L^2/r^2+\epsilon) = 0$.

For timelike geodesics $\epsilon = 1$, for null geodesics, $\epsilon = 0$, for spacelike geodesics $\epsilon = -1$.

The timelike geodesic equation ($\epsilon = 1$), for an unitary mass, may be written in a pseudo-Newtonian Hamiltonian form for compatibility with Newtonian theory.[27]

This comparison suggests to use $(1/2)(dr/d\tau)^2$, instead of $(dr/d\tau)^2$, $\xi = E^2/2$,[28] and a parameter δ.

[27] In weak field, for $r >> 1$ we can neglect the term $2GML^2/(c^4r^3)$, specific to relativity, listed in this equation. Within this assumption, let us compute the geodesic equation by using this equation, by using a method similar to that used in Newtonian theory. For that, one multiplies the equation by $(d\tau/d\varphi)^2 = r^4/L^2)$ and then one set $u = 1/r - GM/L^2$, after simplifying one get $1/r = (GM/L^2)(1 + \sqrt{1+(E^2-1)L^2/GM^2}\cos\varphi)$ which is the solution of the Newtonian theory if we set $E^2 - 1 = 2\xi$, where ξ is the Newtonian energy.

[28] See also [1] p.209-210. In equation (1), 1/2, is the rest mass energy, in general relativity. A constant in a potential, does not change the dynamic. The Newtonian Hamiltonian is: $E = (1/2)((dr/dt)^2 + r^2(d\varphi/dt)^2) - GM/r = (1/2)((dr/d\varphi)^2 + r^2)(C^2/r^4) - GM/r$ with $C = r^2(d\varphi/dt)$.

With these assumptions, on the geodesic, the conserved total energy is ξ, the radial kinetic energy, depending only on the derivative of r, is $(1/2)(dr/d\tau)^2$ and we call *V(r)* a radial potential depending only on r, for a given value of the conserved angular momentum $L = r^2 d\varphi/d\tau$.

$$\frac{\xi}{c^2} = \frac{E^2}{2c^4} = \frac{1}{2}\frac{dr^2}{c^2 d\tau^2} + V(r) \Rightarrow V(r) = \frac{1}{2} - \frac{GM}{c^2 r} + \frac{L^2}{2c^2 r^2} - \delta\frac{GML^2}{c^4 r^3} \tag{42}$$

[29] For $\delta = 0$, we get the classical equation of the Newtonian mechanics. The first term is a constant, the second term is the Newtonian gravitational potential and the third term is a contribution of the angular momentum that takes the same form in Newtonian theory and in general relativity for:

$$L \approx C \Rightarrow L^2 \approx C^2 \tag{43}$$

The conserved energy is E on the geodesic, but the potential *V(r)*, in this Newtonian form responds to $E^2/2$.

Some curves *V(r)* for different values of L will be represented on a diagram, showing that extrema of these curves which correspond to circular orbits will define the corresponding value of r_c and its associated potential $V(r_c)$ without needing to define the energy which is then defined by equation (42).

But, when the orbits are not circular, we have to take into a account the energy, which will correspond to the potential $V(r_E)$ associated, for instance for a bounded orbit, at its perihelion and at its aphelion. This potential $V(r_E)$ will be, per equation (42) equal to the energy as for the extrema $dr/d\tau = 0$.

D Relation between the angular momentum *L* in spacetime and that of its counterpart *l* in space

D.1 Dynamic in space

The space's geodesic equation, where $l = r^2 d\varphi/d\sigma$ is:

$$\frac{dr^2}{d\sigma^2} = (1 - \frac{2GM}{r})(1 - \frac{l^2}{r^2}) \tag{44}$$

We will write this equation in the same Newtonian Hamiltonian form than the relativistic equation described in equation (42), but with $E = 0$. [30] and with a radial potential called *W(r)* instead of *V(r)*, this gives:[31]

$$\frac{E^2}{2} = 0 = \frac{dr^2}{2d\sigma^2} + W(r) \Rightarrow W(r) = -\frac{1}{2} + \frac{GM}{c^2 r} + \frac{l^2}{2r^2} - \delta\frac{GMl^2}{c^2 r^3} \tag{45}$$

D.2 Comparing dynamic in space and dynamic in spacetime

Comparing *V(r)* in equation (42) and *W(r)* in equation (45) shows that if we assume $l^2 = -L^2$, we get: [32]

$$W(r) = -V(r) \tag{46}$$

As the dynamic of such geodesic, around a central mass, is ruled by this radial potential [33], this implies that the dynamic of the space geodesic will be opposite of that of the geodesic in spacetime. Therefore as *V(r)* rules a dynamic corresponding to an attracting phenomenology, *W(r)* drives a dynamic corresponding to a repelling phenomenology.[34]

D.3 The dynamic in spacetime and space illustrated by diagrams

On the figure 2a, for a particle of unit mass with an angular momentum equal to 4, ($L^2 = 16$), a maximum potential of $V(r) = V_0 = E^2/2 = +0,48$, (in units where $GM = 1$ and c = 1), the Newtonian bounded orbit,

[29] We resume c as we will have to compare this equation with its counterpart in space. One can check that all parameters are dimensionless.

[30] In space where there is no time, there is no energy as energy is the physical appearance of time.

[31] We will resume the celerity of light c for dimensional comparison with equation (1). One will be able to check that, as in equation (1), all parameters are dimensionless. Therefore the comparison will be consistent.

[32] This is a consequence of the relation $ds^2 = -d\sigma^2$ when $E = 0$.

[33] In equation (4), the energy $E^2 = 0$ instead of $E^2 \neq 0$ in equation (1), but a constant in such equation does not modify the dynamic.

[34] This can be checked at the Newtonian limit: We get hyperbolas.

C:/Users/jacques/Documents/precession-3/potential.jpg an ellipse in this example, corresponds to the segment BHC of the curve W_1 between perihelion B and aphelion C.

Its counterpart, with same angular parameter, in same units, in spacetime, in relativity is represented by the segment AGD of the curve V1,(EFAGD) between perihelion A and its aphelion D. We notice that for a same angular momentum, aphelion, perihelion, of the Newtonian geodesic and the relativistic geodesic are different. As the potential is maximum at the extremities of this segments A and D, [35] in relativity, and in their counterpart's B and C, in Newtonian gravity, they are turn-back points. This means that, while a test particle is orbiting around the central mass, its image, a point on this curve, will move forth and back, on these segments of curves.

Applying these arguments to Figure 2b, in space, shows that in Newtonian theory, there is no orbit with bounded limits (no well of potential).

In relativity, in space where the energy E is equal to zero, this is different, as a well of potential exists, a bounded orbit exists, between A and C as shown on the figure 1b, in our example.

D.4 Full derivation of the precession in space

The derivation of the solution in this document is purely formal. It was independent of any specific hypothesis about the angular momentum l which can be represented by a real number or a complex number. If we assume that L (angular momentum in spacetime) is represented by a real number, this implies that all parameters of the equations, with $l = L$, would be real. But, in space, we showed that we have to set $l^2 = -L^2/c^2$ which implies that $l = \pm i.L/c$. The consequence is that some parameters including l will be no longer real numbers but complex numbers.

This looks embarrassing as the solution and the precession, which are a physical parameters, must be real numbers. But we will see that, when deriving the solution, a full cancellation of non real terms will occur, leaving only a real terms in the solution, as expected.

Setting $c = 1$ again for simplifying the calculation and inserting $l^2 = -L^2$ in equation (7) yields:

$$\frac{du^2}{u^4 d\varphi^2} = (1 - 2GMu)(\frac{1}{u^4})(\frac{-1 - u^2L^2}{L^2}) \Rightarrow d\varphi = \frac{\pm du}{\sqrt{(1 - 2GMu)(-\frac{1}{L^2} - u^2)}} \tag{47}$$

The solution given by mathematica is:[36]

$$\frac{\varphi}{2} = -\sqrt{\frac{L}{L + i.2GM}}Elliptic_F(\arcsin\sqrt{\frac{1}{2}(1 - i.Lu)}, \frac{4GM}{2GM - iL}) \tag{48}$$

therefore

$$A^2 = \frac{4GM}{2GM - i.L} = (\frac{i.2GM}{L})\frac{2}{\frac{i.2GM}{L} + 1}, K = -\sqrt{\frac{i.L}{i.L - 2GM}} = -\sqrt{\frac{1}{1 + \frac{i.2GM}{L}}} \tag{49}$$

For $Elliptic_K$, we get:

$$\frac{\varphi}{2} = -\sqrt{\frac{1}{1 + \frac{i.2GM}{L}}}Elliptic_K(\frac{i.2GM}{L}\frac{2}{\frac{i.2GM}{L} + 1}) \tag{50}$$

Let us set $2GM/L = m^2$, where, as L and GM are assumed to be real, m^2 is a real-valued parameter. With $l = iL$, and as $k^2 = 2GM/l$ gives $k^2 = i.m^2$, equation (49) becomes:

$$A^2 = \frac{2.k^2}{1 + k^2} = \frac{2.i.m^2}{1 + i.m^2}, K = -\sqrt{\frac{1}{1 + k^2}} = -\sqrt{\frac{1}{1 + i.m^2}} \Rightarrow \frac{\varphi}{2} = -\sqrt{\frac{1}{1 + i.m^2}}Elliptic_K(\frac{2i.m^2}{1 + i.m^2}) \tag{51}$$

The second part of equation (51) giving $\varphi/2$ in space, can be evaluated by the polynomial $P(k^2)$ given by equation (32), by using the value of k^2 in space which is $k^2 = i.m^2$.

We get:

$$\frac{\varphi}{2} = P(k^2) = P(i.m^2) = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(i.m^2)^{2n} = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(-m^4)^n = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(-\frac{(2GM)^2}{(cL)^2})^n \tag{52}$$

[35] Between them there is a well of potential.

[36] Line of command: $FullSimplify[Integrate[1/(Sqrt[(1 - a.u)(-1/l^2 - u^2)])]]$, with $a = 2GM$.

D.5 Numerical value of the polynomial

The polynomial $P(k^2)$ for $k^2 = i.m^2$ is given below up to $n = 10$. [37]

$$P(i.m^2) = -\frac{\pi}{2}(1 - \frac{3}{16}(m^4) + \frac{105}{1024}(m^8) - \frac{1155}{16384}(m^{12}) + \frac{225225}{4194304}(m^{16}) - \frac{2909907}{67108864}(m^{20})) \tag{53}$$

Where $m^4 = (2GM/cL)^2$ is a dimensionless, real parameter.

D.6 In weak field, the approximation of the precession in space, yields half of the result of that in spacetime

In weak field when A^2 is small, let us use equation (52) with B_n defined in equation (25), limited, at third order.

$$\frac{\varphi}{2} \approx -\frac{\pi}{2}\sum_{n=0}^{n=1} B_2 n(-m^4)^n, B_2 n = \frac{(4n)!}{2n!^2}(-1^{2n})(2^{-4n}) \sum_{j=0}^{j=2n}(-\frac{1}{2})^j \binom{2n}{j}\binom{2j}{j} \tag{54}$$

As it is obvious that B_0, for $n = 0$, is equal to 1, we just have to compute $B_2, (n = 1)$

$$B_2 = \frac{4!}{2!^2}(-1^2)(2^{-4})\sum_{j=0}^{j=2}(-\frac{1}{2})^j \binom{2}{j}\binom{2j}{j} = \frac{3}{16} \tag{55}$$

Therefore, the result for $\pi/2$, resuming the celerity of light c, with $(k^2)^2 = -m^4 = -(2GM/cL)^2$ which is dimensionless parameter, is:

$$\frac{\varphi}{2} \approx -\frac{\pi}{2}(1 + \frac{3(-m^4)}{16}) = -\frac{\pi}{2}(1 - \frac{3(2GM)^2}{16c^2L^2}) = -\frac{\pi}{2}(1 - \frac{3(GM)^2}{4c^2L^2}) \Rightarrow \Delta\varphi_{(\psi=2\pi)} = \frac{3\pi(GM)^2}{c^2L^2} \tag{56}$$

The final result, a positive precession which is half of the well-known result of the precession in spacetime.

D.7 Spatial precession of the planet Mercury, computed with this method

Mercury orbital period: 0.2408467 yr, semi-major axis $a = 57,909,176$ km, eccentricity $e = 0.20563069$, solar mass:1.98855×10^{30} kg, $G = 6.67384 \times 10^{-11}$, $c = 299792458 m/s$. In Newtonian mechanics, for such elliptic orbit, the angular momentum C is defined by $C^2 = GM.a(1 - e^2)$. With the parameters of the planet Mercury listed above we get:

$C^2 = 6.67384 \times 10^{-11} \times 1.98855 \times 10^{30} \times 5.7909176 \times 10^{10} \times (1 - 0.20563069^2) = 7.36032 \times 10^{30}$

We will use equation (56) for computing the precession, with $l^2 \approx -C^2$, therefore we get:

$\Delta\varphi/2 \approx \pi/2(\frac{3\times(GM)^2}{4c^2C^2}) = 3.136665 10^{-8}$ rad for $\pi/2$. [38]

For getting $\Delta\varphi$ for 2π, we will multiply the above value of $\Delta\varphi/2$ for $\pi/2$, first by 4 for getting $\Delta\varphi/2$ for 2π then by 2 for getting $\Delta\varphi$ for 2π. In 100 years, there are 415.2 orbits and there are π radians in 180 degrees.

The result of the precession per century, in arc-seconds, is then given by:

$\Delta\varphi = 2 \times (3.136665 10^{-8} \times ((180/\pi) \times 3600) \times (4) \times 415.202) \approx 21.4903$ arc-seconds per century. $\Delta\varphi = 42.9806/2$ arc-seconds per century.[39]

D.8 Formal solution with the geometrical parameters of the Newtonian ellipse

In weak field the phenomenology of precession is often modelized by a slowly-rotating Newtonian ellipse of semi-major axis a and eccentricity e, whose angular momentum C^2 is defined by:

$$C^2 = GMa(1 - e^2) \tag{57}$$

Then, using equation (43)

$$L^2 = C^2 = GMa(1 - e^2) \tag{58}$$

[37] Computed by using mathematica, line of command : Bn = FullSimplify [((2n)!/(n!)^2)(2^{-2n}((-1)^n) Sum [((-1/2)^j) Binomial [n, j] Binomial [2j, j],(j, 0, n)]], for $n = 2, 4, 6, 8, 10$.

[38] Computed with mathematica 4.

[39] For Mercury, we may use the Newtonian parameters. It is half of the well-known result in spacetime (42.98 arc-seconds) obtained by other methods.

We will get the precession by inserting, in equation (56), the value of L, given by equation (58). For a full orbit, $\psi = 2\pi$, this gives:

$$\frac{\varphi(2\pi, A^2)}{2} \approx -2\pi(1 - \frac{3(GM)^2}{4c^2GMa(1-e^2)}) = -2\pi(1 - \frac{3GM}{4c^2a(1-e^2)} \tag{59}$$

This is the value for $\varphi/2$. For getting φ for a full orbit, we have to multiply by 2. We get:

$$\varphi(2\pi, A^2) \approx -4\pi(1 - \frac{3GM}{4c^2a(1-e^2)}) \Rightarrow \Delta\varphi = \varphi(2\pi, A^2) - (-4\pi) \approx \frac{3\pi GM}{c^2a(1-e^2)} \tag{60}$$

We get half of the well-known formula for $\Delta\varphi$ which defines the precession per orbital period. The precession is positive, as expected in spacetime.

E Annex 4: Deflection of light

E.1 Example: deflection maximum of light by the Sun (weak field)

Radius of the Sun $l = 6.957 \times 10^5$ km. Half of Schwarzschild radius : $GM = 1.4765$ km. Equation (11) shows that for the perihelion, $r = l, \theta = \pi/2$, at infinity ($r = \infty$), $\theta = \pi/4$

By setting $GM = 1$ we get $l = 4.711818 \times 10^5$ and $\frac{GM}{l} = 2.12232306 \times 10^{-6}$. With this values [40]:

$$\sqrt{\frac{l}{l+2GM}} = 0.999997877, \frac{4GM}{l+2GM}) = 8.48925621 \times 10^{-6}, Elliptic_K(\frac{4GM}{l+2GM}) = 1.5707996605339 \tag{61}$$

$$Elliptic_F(\frac{\pi}{4}, \frac{4GM}{l+2GM}) = 0.7853987691 \tag{62}$$

The precession of $\varphi/2$, for a half geodesic, is defined by:

$$\Delta\frac{\varphi}{2} = [(\sqrt{\frac{l}{l+2GM}})(Elliptic_K(\frac{4GM}{l+2GM}) - Elliptic_F(\frac{\pi}{4}, \frac{4GM}{l+2GM}))] - \frac{\pi}{4} \tag{63}$$

With the numerical values computed previously this yields:

$$\Delta\frac{\varphi}{2} = 0.999997877(1.5707996605339 - 0.7853987691) - 0.7853981634 = 1.06062 \times 10^{-6} \tag{64}$$

For $\varphi/2$, and a full orbit, we have to multiply by $2 \times 2 = 4$. This yields:

$$\Delta\varphi = 4.24249 \times 10^{-6} rd = 0.875076" \tag{65}$$

This is half of the well known result (1.75 "). [41]

F Acknowledgments

I am grateful to the professor Jean Jacques Szczeciniarz [42] for his advices and his support, for this article. I thank Denis Mercier [43] for his kind support on some mathematical issues. I thank Olivier Darrigol [44] for providing references to analytic solutions and Marc Lachièze-Rey [45] for his comments and suggestions.

[40] The result of the elliptic integrals are obtained by using mathematica.
[41] in Newtonian theory we get the same result.
[42] Jean Jacques Szczeciniarz is professor of philosophy and director of the department- History and philosophy of sciences -at the université Paris-Diderot, Laboratoire SPHERE.
[43] Denis Mercier, Université de Valenciennes, Laboratoire (Lamav).
[44] Olivier Darrigol, Directeur de recherches - CNRS/Université Denis Diderot.
[45] Marc Lachièze-Rey, Directeur de recherches - CNRS/Université Denis Diderot.

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